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THE EFFECT OF TRANSVERSE PRESSURE ON THE STABILITY OF A PLATE

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The stability problem of a centrally compressed infinite plate is solved with allowance for the transverse normal deformation caused by uniform load for various boundary conditions at the edges. The linearized nonlinear equations of elastic deformation of thin plates taking into account transverse shear and transverse normal deformation are used. The obtained critical loads are compared with existing solutions.

Key words: *stability, elasticity, plate, Legendre polynomials.*

In [1], the basic three-dimensional nonlinear elasticity problem is reduced to a sequence of two-dimensional problems using a Legendre polynomial expansion. Nonlinear and corresponding linearized equations governing the elastic deformation of plates with arbitrary boundary conditions on the surfaces were obtained. The first-approximation linearized equations for thin plates that take into account transverse shear and transverse normal deformation are given in [2].

To illustrate the efficiency of using these equations, we solve the stability problem of a centrally compressed infinite plate. In addition to the axial force, a uniform normal load of constant magnitude and direction is applied to the surfaces.

Loads applied to a plate that have constant direction and magnitude belong to the class of conservative or dead loads. In this case, the loss of stability can be treated as static instability [3, 4]. To determine the critical load, we use a static method (Euler's method). The critical loads are obtained for five versions of boundary conditions at the edges similar to those adopted in classical stability theory for centrally compressed bars [5, 6].

For some particular loading cases considered in this paper, a similar problem was solved using other methods [7, 8].

Using an energy method, Kerr and Tang [7] solved the stability problem of a compressed band subject to the following conditions at the edges: one edge is rigidly clamped and the other edge is allowed to slide over the die surface. Guz' [8] solved the stability problem of a simply supported band under uniform compressing dead load. The problem was considered using three-dimensional linearized elasticity theory for finite and small subcritical strains [9].

We consider an infinitely long plate of width l and thickness $2h$. Let x_k be a Cartesian coordinate system with basis $\boldsymbol{e}_k$. The x_1 axis ($x_1 \in [0, l]$) is directed along the width of the plate, the x_2 axis along the length, and the x_3 axis ($x_3 \in [-h, h]$) along the thickness. Assuming plane-strain conditions, we solve the stability problem of the plate compressed along the x_1 axis by a force of intensity p applied to the edges ($x_1 = 0, l$). The plate surfaces ($x_3 = \pm h$) are subjected to a uniformly distributed transverse load of intensity q which has constant direction and magnitude.

We introduce a Lagrangian coordinate system ξ^k with covariant basis $\boldsymbol{g}_k$. In the undeformed state,

$$x_\alpha = \xi^\alpha \quad (\alpha = 1, 2), \qquad x_3 = h\xi^3, \qquad \mathring{\boldsymbol{g}}_\alpha = \boldsymbol{e}_\alpha, \qquad \mathring{\boldsymbol{g}}_3 = h\boldsymbol{e}_3. \tag{1}$$

Here and below, the superimposed circle denotes the undeformed state.

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Let $\boldsymbol{u}$ be the displacement vector in the subcritical state, $\hat{\boldsymbol{t}}^i = \mathring{J}\tau^{ij}\boldsymbol{g}_j$ be quantities that characterize the subcritical stress state, τ^{ij} be the covariant components of the second Piola–Kirchhoff stress tensor [4], and $J = \boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)$ be the Jacobian of The coordinate transformation ξ^k. The covariant basis of the subcritical state $\boldsymbol{g}_k$ is linked to the displacement vector $\boldsymbol{u}$ by the relations

$$\boldsymbol{g}_i = \mathring{\boldsymbol{g}}_i + \boldsymbol{u}_{,i}.$$

In addition to the subcritical state, we consider the perturbed state characterized by the perturbed quantities

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \Delta\boldsymbol{u}, \qquad \tilde{\hat{\boldsymbol{t}}}^i = \hat{\boldsymbol{t}}^i + \Delta\hat{\boldsymbol{t}}^i, \qquad \tilde{\boldsymbol{g}}_i = \boldsymbol{g}_i + \Delta\boldsymbol{g}_i, \qquad \Delta\boldsymbol{g}_i = \Delta\boldsymbol{u}_{,i}. \tag{2}$$

It follows from (1) that $\xi^3 \in [-1, 1]$. The perturbations $\Delta\boldsymbol{u}$ and $\Delta\hat{\boldsymbol{t}}^i$ in (2) are written as series in Legendre polynomials $P_k(\xi^3)$:

$$\Delta\boldsymbol{u} = \sum_{k=0}^{\infty} [\Delta\boldsymbol{u}]^k P_k, \qquad \Delta\hat{\boldsymbol{t}}^i = \sum_{k=0}^{\infty} [\Delta\hat{\boldsymbol{t}}^i]^k P_k. \tag{3}$$

Here $[\Delta\boldsymbol{u}]^k$ and $[\Delta\hat{\boldsymbol{t}}^i]^k$ the expansion coefficients, which for plane strain depend on one coordinate ξ^1:

$$[\Delta\boldsymbol{u}]^k = \frac{1+2k}{2} \int_{-1}^{1} \Delta\boldsymbol{u} P_k \, d\xi^3, \qquad [\Delta\hat{\boldsymbol{t}}^i]^k = \frac{1+2k}{2} \int_{-1}^{1} \Delta\hat{\boldsymbol{t}}^i P_k \, d\xi^3.$$

We use the following assumptions:

— the subcritical state is governed by geometrically linear equations and in formulas (2) one should set

$$\boldsymbol{g}_i \cong \mathring{\boldsymbol{g}}_i; \tag{4}$$

— the subcritical state is uniform:

$$\tau^{11} = -p, \qquad \tau^{33} = -q/h^2, \qquad \tau^{ij} = 0 \quad (i \neq j); \tag{5}$$

— the plate material is isotropic.

Following the results of [2], the first approximation for the quantities $\Delta\hat{\boldsymbol{t}}^i$ characterizing the perturbations in the stresses in (2) is obtained using truncated series (3):

$$\begin{gathered} 2h(\boldsymbol{e}_1 \cdot \Delta\hat{\boldsymbol{t}}^1) \cong N + 3MP_1/h, \qquad (\boldsymbol{e}_3 \cdot \Delta\hat{\boldsymbol{t}}^3) \cong p_0 + p_\Delta P_1, \\ 2h(\boldsymbol{e}_1 \cdot \Delta\hat{\boldsymbol{t}}^3) \cong Q + 2hq_\Delta P_1 + (2hq_0 - Q)P_2, \qquad 2h(\boldsymbol{e}_3 \cdot \Delta\hat{\boldsymbol{t}}^1) \cong F, \\ q_\Delta = (q^+ - q^-)/2, \qquad q_0 = (q^+ + q^-)/2, \\ p_\Delta = (p^+ - p^-)/2, \qquad p_0 = (p^+ + p^-)/2. \end{gathered} \tag{6}$$

In formulas (6), the following notation is adopted:

$$\begin{gathered} N = 2h\boldsymbol{e}_1 \cdot [\Delta\hat{\boldsymbol{t}}^1]^0, \qquad F = 2h\boldsymbol{e}_3 \cdot [\Delta\hat{\boldsymbol{t}}^1]^0, \qquad Q = 2h\boldsymbol{e}_1 \cdot [\Delta\hat{\boldsymbol{t}}^3]^0, \\ M = (2/3)h^2\boldsymbol{e}_3 \cdot [\Delta\hat{\boldsymbol{t}}^1]^1, \qquad p^\pm = \boldsymbol{e}_3 \cdot \Delta\hat{\boldsymbol{t}}^3\Big|_{\xi^3=\pm1}, \qquad q^\pm = \boldsymbol{e}_1 \cdot \Delta\hat{\boldsymbol{t}}^3\Big|_{\xi^3=\pm1}. \end{gathered} \tag{7}$$

Under plane-strain conditions, the equilibrium equations for the perturbations in the projections onto the x_1 and x_3 axes become

$$\boldsymbol{e}_1 \cdot (\Delta\hat{\boldsymbol{t}}^1_{,1} + \Delta\hat{\boldsymbol{t}}^3_{,3}) = 0, \qquad \boldsymbol{e}_3 \cdot (\Delta\hat{\boldsymbol{t}}^1_{,1} + \Delta\hat{\boldsymbol{t}}^3_{,3}) = 0. \tag{8}$$

Integrating (8) across the thickness, we obtain the equilibrium equations for the perturbations in the forces, moments, and surface loads at the surfaces (7):

$$N_{,1} + 2hq_\Delta = 0, \qquad M_{,1} - hQ + 2h^2 q_0 = 0, \qquad F_{,1} + 2hp_\Delta = 0. \tag{9}$$

Assuming that the subcritical state (4) is linear and following [1], we approximate Hooke's law for the perturbations $\Delta\hat{\boldsymbol{t}}^i$ as

$$\Delta\hat{\boldsymbol{t}}^i \cong \mathring{J}\tilde{C}^{ijmn}(\mathring{\boldsymbol{g}}_m \cdot \Delta\boldsymbol{G}_n)\mathring{\boldsymbol{g}}_j. \tag{10}$$

In (10), the approximations $\Delta\boldsymbol{G}_n$ of the perturbations in the vectors of the covariant basis $\Delta\boldsymbol{g}_n$ have the form

$$\Delta\boldsymbol{g}_n \cong \Delta\boldsymbol{G}_n, \qquad \Delta\boldsymbol{G}_1 = \Delta\boldsymbol{U}'_{,1}, \qquad \Delta\boldsymbol{G}_2 = 0, \qquad \Delta\boldsymbol{G}_3 = \Delta\boldsymbol{U}''_{,3}. \tag{11}$$

The vectors $\Delta\boldsymbol{U}'$ and $\Delta\boldsymbol{U}''$ in (11) are approximations of the perturbations in the displacement vector $\Delta\boldsymbol{u}$; they are used to calculate the derivatives with respect to the coordinates ξ^1 and ξ^3, respectively. These approximations differ in the number of terms retained in the Legendre polynomial expansions (3):

$$\begin{gathered}\Delta\boldsymbol{u} \cong \Delta\boldsymbol{U}' = \boldsymbol{e}_1(u + \psi P_1) + \boldsymbol{e}_3 v,\\ \Delta\boldsymbol{u} \cong \Delta\boldsymbol{U}'' = \Delta\boldsymbol{U}' + \boldsymbol{e}_1([u]^2 P_2 + [u]^3 P_3) + \boldsymbol{e}_3([v]^1 P_1 + [v]^2 P_2).\end{gathered} \tag{12}$$

For an isotropic medium, we have

$$\tilde{C}^{ijmn} = \lambda \mathring{g}^{ij}\mathring{g}^{mn} + \mu(\mathring{g}^{im}\mathring{g}^{jn} + \mathring{g}^{in}\mathring{g}^{jm}) + \tau^{in}\mathring{g}^{mj}$$

(λ and μ are the Lamé parameters).

By virtue of (1), the contravariant components of the metric tensor $\mathring{g}^{ij}$ of the coordinate system ξ^k for the undeformed state are given by

$$\mathring{g}^{11} = 1, \qquad \mathring{g}^{22} = 1, \qquad \mathring{g}^{33} = 1/h^2, \qquad \mathring{g}^{ij} = 0 \quad (i \neq j). \tag{13}$$

It follows from (5) and (13) that the components $\tilde{C}^{ijmn}$ vanish except the following:

$$\begin{gathered}\tilde{C}^{1111} = \lambda + 2\mu - p, \qquad \tilde{C}^{1133} = \tilde{C}^{3311} = \lambda/h^2,\\ \tilde{C}^{1331} = (\mu - p)/h^2, \qquad \tilde{C}^{3113} = (\mu - q)/h^2,\\ \tilde{C}^{1313} = \tilde{C}^{3131} = \mu/h^2, \qquad \tilde{C}^{3333} = (\lambda + 2\mu - q)/h^4.\end{gathered} \tag{14}$$

We insert formulas (11)–(14) into (10). Substitution of the resulting expressions for the perturbations $\Delta\hat{\boldsymbol{t}}^i$ into (7) yields two groups of equalities. The first group contains equations for the perturbations of the forces and moments:

$$\begin{gathered}N/(2h^2) = (\lambda + 2\mu - p)u_{,1} + \lambda[v]^1/h, \qquad 3M/(2h^3) = (\lambda + 2\mu - p)\psi_{,1} + 3\lambda[v]^2/h,\\ Q/(2h) = \mu v_{,1} + (\mu - q)(\psi + [u]^3)/h, \qquad F/(2h^2) = (\mu - q)v_{,1} + \mu(\psi + [u]^3)/h,\end{gathered} \tag{15}$$

and the second group contains equations for the perturbations of the external forces on the surfaces:

$$\begin{gathered}p_0 = \lambda u_{,1} + (\lambda + 2\mu - q)[v]^1/h, \qquad p_\Delta = \lambda\psi_{,1} + 3(\lambda + 2\mu - q)[v]^2/h,\\ q_\Delta = 3(\mu - q)[u]^2/h, \qquad q_0 - Q/(2h) = 5(\mu - q)[u]^3/h.\end{gathered} \tag{16}$$

The set of the equilibrium equations (9) and Hooke's relations (15) and (16) is the system of linearized first-approximation equations for the nonlinear problem of elastic deformation of thin plates with allowance for transverse shear and transverse normal deformation.

In the static method of solving stability problems, the perturbations of the external forces are set equal to zero, i.e., the possible equilibrium states are considered. Consequently, on the surfaces,

$$q_0 = 0, \qquad q_\Delta = 0, \qquad p_0 = 0, \qquad p_\Delta = 0. \tag{17}$$

The system of four algebraic equations (16) and (17) for the four unknown functions $[u]^2$, $[u]^3$, $[v]^2$, and $[v]^3$ has the solution

$$[u]^2 = 0, \qquad [u]^3 = -\frac{1}{6}\Big(\psi + \frac{\mu}{\mu - q}\,hv_{,1}\Big), \qquad [v]^1 = -\frac{\lambda}{\lambda + 2\mu - q}\,hu_{,1}, \qquad [v]^2 = -\frac{\lambda}{\lambda + 2\mu - q}\,h\psi_{,1}. \tag{18}$$

From (15) and (18), it follows

$$\begin{gathered}N = 2h^2\Big(\lambda + 2\mu - \frac{\lambda^2}{\lambda + 2\mu - q} - p\Big)u_{,1}, \qquad M = \frac{2}{3}\,h^3\Big(\lambda + 2\mu - \frac{\lambda^2}{\lambda + 2\mu - q} - p\Big)\psi_{,1},\\ Q = \frac{5}{3}\,h\Big(\mu v_{,1} + (\mu - q)\frac{\psi}{h}\Big), \qquad F = 2h^2\Big(\Big(\mu - p - \frac{\mu^2}{6(\mu - q)}\Big)v_{,1} + \frac{5}{6}\,\mu\,\frac{\psi}{h}\Big).\end{gathered} \tag{19}$$

We insert expressions (17) and (19) into the equilibrium equations (9). As a result, we have a linear system of three ordinary differential equations for the three unknown functions u, v, and ψ:

$$u_{,11} = 0,$$
$$\Big(\frac{(\lambda+2\mu)^2-\lambda^2-q^2}{\lambda+2\mu-q} - p - q\Big)\psi_{,11} - \frac{5}{2h}\Big((\mu-q)\,\frac{\psi}{h} + \mu v_{,1}\Big) = 0, \tag{20}$$
$$\psi_{,1} + \frac{6}{5}\Big(1 - \frac{p}{\mu} - \frac{\mu}{6(\mu-q)}\Big)hv_{,11} = 0.$$

Eliminating ψ, we obtain the equation for the deflection v

$$v_{,1111} + \alpha^2 v_{,11}/h^2 = 0, \qquad \alpha^2 = B/A, \tag{21}$$

in which

$$A = (\gamma - \bar{p})(5/6 - \bar{p})/(1-\varepsilon)^2, \qquad B = 5\bar{p}/2,$$

$$\varepsilon = q/\mu, \qquad \gamma = \beta(2-\varepsilon)^2/(\beta + (1-\varepsilon)), \qquad \beta = (\lambda+\mu)/\mu, \qquad \bar{p} = (q + p(1-\varepsilon))/\mu.$$

The general solution of the ordinary differential equation (21) is given by

$$v = C_1 + C_2 x + C_3 \sin(\alpha x/h) + C_4 \cos(\alpha x/h). \tag{22}$$

Substituting (22) into system (20), we obtain the general solution for the function ψ:

$$\psi = -(hC_2 + \alpha(1 - 6\bar{p}/5)(C_3\cos(\alpha x/h) - C_4 \sin(\alpha x/h)))/(1-\varepsilon). \tag{23}$$

The last relation in (21) is a quadratic equation for $\bar{p}$:

$$(\gamma - \bar{p})(5/6 - \bar{p})\bar{\alpha}^2 = 5\bar{p}/2, \qquad \bar{\alpha} = \alpha/(1-\varepsilon). \tag{24}$$

Using the smaller root of the quadratic equation (24), we find the limit load:

$$\bar{p}(\bar{\alpha}) = 2\gamma\bar{\alpha}^2/(3\phi(\bar{\alpha})), \qquad \phi(\bar{\alpha}) = 1 + a_1\bar{\alpha}^2 + \sqrt{1 + 2a_1\bar{\alpha}^2 + a_2^2\bar{\alpha}^4}\,,$$
$$a_1 = (5+6\gamma)/15, \qquad a_2 = (5-6\gamma)/15. \tag{25}$$

In formulas (22), (23), and (25), the unknown functions v and ψ and the limit load $\bar{p}$ depend on the constants C_i and α, which are determined from the boundary conditions at the plate edges. By analogy with the classical case [5], five versions of the boundary-value problem are possible (see Fig. 1). By the classical case is meant the solution of the stability problem of a plate (rod) for $q = 0$ obtained using the Kirchhoff–Love hypotheses (Euler's solution).

Version I. The edges are simply supported:

$$v = 0, \qquad M = 0 \qquad \text{for} \quad x = 0, l. \tag{26}$$

Using the expression for the perturbation of the moment M in (19), we substitute (22) and (23) into the boundary conditions (26) and obtain the following homogeneous system of four linear algebraic equations:

$$C_1 + C_4 = 0, \qquad C_1 + C_2 l + C_3 \sin(\alpha l/h) + C_4 \cos(\alpha l/h) = 0,$$
$$C_4 = 0, \qquad C_3 \sin(\alpha l/h) + C_4\cos(\alpha l/h) = 0. \tag{27}$$

The system of homogeneous equations (27) has a nontrivial solution if its determinant Δ_1 (the subscript refers to the version number) vanishes:

$$\Delta_1 = l \sin(\alpha l/h) = 0.$$

Using the minimum positive root of this equation α, we obtain

$$\alpha_1 = \pi h/l. \tag{28}$$

The solution of system (27) is determined with accuracy up to an arbitrary constant C and is given by $C_1 = C_2 = C_4 = 0$ and $C_3 = C$. The corresponding expressions for v and ψ are determined from formulas (22) and (23):

$$v_1 = C\sin(\alpha_1 x/h), \qquad \psi_1 = -C\bar{\alpha}_1(1 - 6\bar{p}_1/5)\cos(\alpha_1 x/h). \tag{29}$$

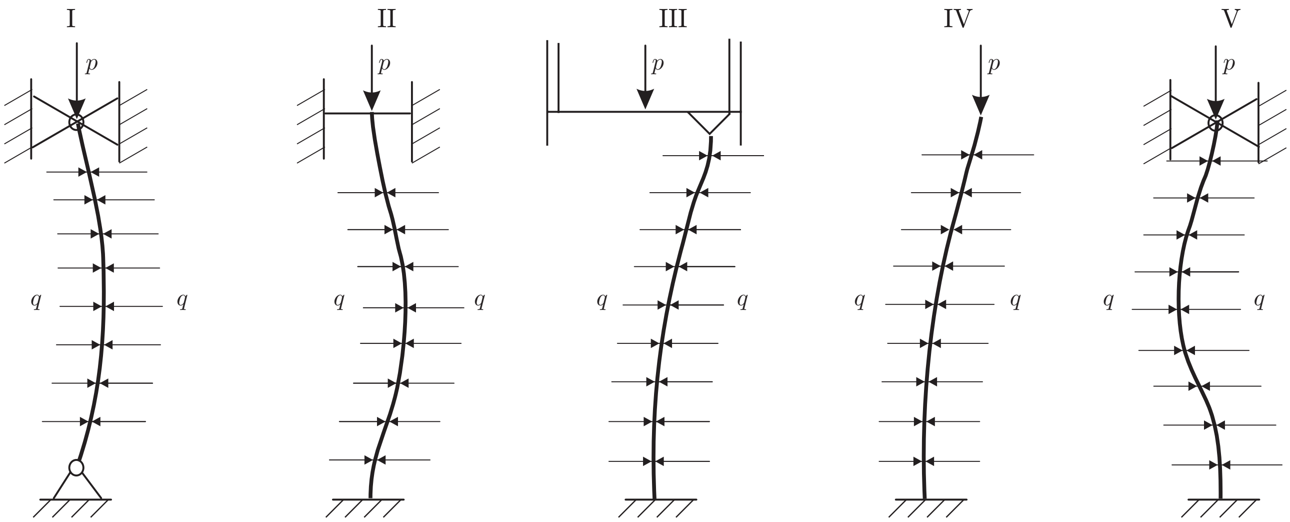

Fig. 1

Expressions (29) define the buckling mode. The deflection v does not depend on the transverse pressure q and coincides with the classical case.

The limit load $\bar{p}_1 = \bar{p}(\bar{\alpha}_1)$ is calculated by substituting (28) into (25).

For the other versions of boundary conditions at the edges, the reasoning is similar to that in version I. Therefore, omitting intermediate calculations, we consider only the main results.

Version II. The edges are rigidly clamped:

$$v = 0, \qquad \psi = 0 \qquad \text{for} \quad x = 0, l. \tag{30}$$

Substitution of (22) and (23) into (30) yields a system of homogeneous equations, whose determinant Δ_2 is set equal to zero:

$$\Delta_2 = 2\alpha\Big(1 - \frac{6}{5}\,\bar{p}(\bar{\alpha})\sin\frac{\alpha l}{2h}\Big(2h\sin\frac{\alpha l}{2h} - l\alpha\Big(1 - \frac{6}{5}\,\bar{p}(\bar{\alpha})\Big)\cos\frac{\alpha l}{2h}\Big)\Big) = 0. \tag{31}$$

Equality (31) implies that $\sin(\alpha l/(2h)) = 0$ or $\tan(\alpha l/(2h)) = (1 - 6\bar{p}(\bar{\alpha})/5)\alpha l/(2h)$. In the first case, the minimum positive value α is given by

$$\alpha_2 = 2\pi h/l. \tag{32}$$

The solution of the homogeneous system gives the unknowns C_i with accuracy up to an arbitrary constant C. The corresponding expressions for v and ψ are obtained from formulas (22) and (23):

$$v_2 = C\sin^2(\alpha_2 x/(2h)), \qquad \psi_2 = C\bar{\alpha}_2(1 - 6\bar{p}_2/5)\sin(\alpha_2 x/h). \tag{33}$$

Expressions (33) define the buckling mode. As for version I, the deflection v_2 does not depend on the transverse pressure q and is expressed as in the classical case.

The limit load $\bar{p}_2 = \bar{p}(\bar{\alpha}_2)$ is calculated by substituting (32) into (25).

In the second case, α is determined from the transcendental equation

$$\tan(\alpha l/(2h)) = \alpha l(1 - 6\bar{p}(\bar{\alpha})/5)/(2h),$$

whose approximate solution is given by

$$\alpha = (2h/l)(4.4934 + O(h^2/l^2)).$$

The corresponding limit load is higher than that in the first case and, hence, this case is ignored.

Version III. At the edges, the following conditions are imposed:

$$v = 0, \quad \psi = 0 \quad \text{for } x = 0, \qquad F = 0, \quad \psi = 0 \quad \text{for } x = l. \tag{34}$$

Using the expressions for the perturbation of F from (19) and inserting (22) and (23) into (34), we obtain a system of homogeneous equations, whose determinant Δ_3 is set equal to zero:

$$\Delta_3 = -\alpha^2(1 - 6\bar{p}(\bar{\alpha})/5)^2 \sin(\alpha l/h) = 0. \tag{35}$$

From (35), we infer that the minimum positive value of α is

$$\alpha_3 = \pi h/l. \tag{36}$$

The solution of the homogeneous system gives the unknowns C_i with accuracy up to an arbitrary constant C. The corresponding expressions for v and ψ follow from formulas (22) and (23):

$$v_3 = C\sin^2(\alpha_3 x/(2h)), \qquad \psi_3 = C\bar{\alpha}_3(1 - 6\bar{p}_3/5)\sin(\alpha_3 x/h). \tag{37}$$

Expressions (37) define the buckling mode. As in the previous cases, the deflection v_3 does not depend on the pressure q and is expressed as in the classical case.

The limit load $\bar{p}_3 = \bar{p}(\bar{\alpha}_3)$ is calculated by inserting (36) into (25).

Version IV. At the edges, the following conditions are imposed:

$$v = 0, \quad \psi = 0 \quad \text{for } x = 0, \qquad F = 0, \quad M = 0 \quad \text{for } x = l. \tag{38}$$

Using the expressions for the perturbations of F and M in (19) and substituting (22) and (23) into (38), we obtain a system of homogeneous equations, whose determinant Δ_4 is set equal to zero:

$$\Delta_4 = -\alpha(1 - 6\bar{p}(\bar{\alpha})/5)\cos(\alpha l/h) = 0. \tag{39}$$

From (39) it follows that the minimum positive value of α is given by

$$\alpha_4 = \pi h/(2l). \tag{40}$$

The solution of the homogeneous system gives the unknowns C_i with accuracy up to an arbitrary constant C. The corresponding expressions for v and ψ follow from formulas (22) and (23):

$$v_4 = C\sin^2(\alpha_4 x/(2h)), \qquad \psi_4 = C\bar{\alpha}_4(1 - 6\bar{p}_4/5)\sin(\alpha_4 x/h). \tag{41}$$

Formulas (41) determine the buckling mode. As earlier, the deflection v_4 does not depend on the transverse pressure q and is expressed as in the classical case.

The limit load $\bar{p}_4 = \bar{p}(\alpha_4)$ is calculated by substituting (40) into (25).

Version V. The following conditions are specified at the edges:

$$v = 0, \quad \psi = 0 \quad \text{for } x = 0, \qquad v = 0, \quad M = 0 \quad \text{for } x = l. \tag{42}$$

Using the expression for the perturbation of the moment M from (19) and substituting (22) and (23) into (42), we obtain a system of homogeneous equations and equate its determinant Δ_5 to zero:

$$\Delta_5 = h\cos(\alpha l/h) - l\alpha(1 - 6\bar{p}(\bar{\alpha})/5)\sin(\alpha l/h) = 0. \tag{43}$$

From (43), we obtain the transcendental equation for α_5

$$\tan\,(\alpha l/h) = (\alpha l/h)(1 - 6\bar{p}(\alpha)/5),$$

whose approximate solution is given by

$$\alpha_5 = (h/l)(4.4934 + O(h^2/l^2)) \simeq 4.4934h/l. \tag{44}$$

The solution of the homogeneous system gives the unknowns C_i with accuracy up to an arbitrary constant C. The corresponding expressions for v and ψ are found from formulas (22) and (23):

$$v_5 = C(1 - x/l + \cot\,(\alpha_5 l/h)\sin(\alpha_5 x/h) - \cos(\alpha_5 x/h)),$$
$$\psi_5 = C(h/l)(1 - \cos(\alpha_5 x/h) - \tan\,(\alpha_5 l/h)\sin(\alpha_5 x/h))/(1 - \varepsilon). \tag{45}$$

Expressions (45) define the buckling mode. The deflection v_5 does not depend on the transverse pressure q and is expressed as in the classical case.

The limit load $\bar{p}_5 = \bar{p}(\bar{\alpha}_5)$ is calculated by substituting (44) into (25).

By analogy with the classical formulas, using the concept of normalized length introduced by Yasinskii [6], we write

$$\alpha_i = \pi h/l_i \qquad (i = \overline{1,5}), \tag{46}$$

where l_i is the normalized length: $l_1 = l$, $l_2 = l/2$, $l_3 = l$, $l_4 = 2l$, and $l_5 \simeq 0.7l$. In this case, formula (25) for calculating the limit load $\bar{p}_i$ for all versions of boundary conditions at the edges is written in the same form:

$$\bar{p}_i = 2\gamma\alpha_i^2/(3\phi_i), \qquad \phi_i = \phi(\bar{\alpha}_i) = 1 + \bar{\alpha}_i^2 a_1 + \sqrt{1 + 2\bar{\alpha}_i^2 a_1 + \bar{\alpha}_i^4 a_2^2}\,.$$

Expanding the right side in a series in powers of $\bar{\alpha}_i^2$, with accuracy up to $\bar{\alpha}_i^4$ we obtain

$$\bar{p}_i = 2\gamma\alpha_i^2(1 - \bar{\alpha}_i^2 a_1)/3. \tag{47}$$

Thus, the limit load obtained for the basic case (version I) can be used for the other boundary conditions at the edges by replacing the real width of the plate l by the normalized length l_i.

We next assume that $q \ll \mu$. In this case, $1 \pm \varepsilon \simeq 1$ and the constants in formulas (47) containing ε become

$$\bar{\alpha} \simeq \alpha, \qquad \gamma \simeq \frac{4(\lambda + \mu)}{\lambda + 2\mu} = \frac{2}{1 - \nu}, \qquad a_1 \simeq \frac{17 - 5\nu}{15(1 - \nu)}, \qquad \bar{p} = \frac{q + p}{\mu}$$

(ν is Poisson's ratio).

We write formula (47) in equivalent form

$$(p + q)_i = p_{ei}\Big(1 - \alpha_i^2\,\frac{17 - 5\nu}{15(1 - \nu)}\Big), \tag{48}$$

where $p_{ei} = \alpha_i^2 E/(3(1 - \nu^2))$ is the critical Euler load and E is Young's modulus.

The critical load (48) is determined as the sum of the axial and transverse loads. The axial load p decreases with increase in q. A similar result was obtained in [7] for a particular case of boundary conditions at the edges (version III).

The solution of the stability problem of a simply supported band (version I) under uniform compression by a dead load is given in [8]. The following critical loads p_L and p_N were obtained for the linear and nonlinear subcritical states, respectively:

$$p_L = \frac{1}{2}\,p_{e1}\Big(1 - \alpha_1^2\,\frac{14 - 23\nu + \nu^2}{30(1 - \nu)^2}\Big), \qquad p_N = \frac{1}{2}\,p_{e1}\Big(1 - \alpha_1^2\,\frac{2 + 3\nu}{15(1 - \nu)}\Big). \tag{49}$$

For uniform compression, $p = q$ and formula (48) becomes

$$p_1 = \frac{1}{2}\,p_{e1}\Big(1 - \alpha_1^2\,\frac{17 - 5\nu}{15(1 - \nu)}\Big). \tag{50}$$

A comparison of the critical loads (49) and (50) yields the inequalities

$$p_1 < p_N < p_L \qquad \text{for} \quad 0 \le \nu \le 0.5.$$

As $\alpha \to 0$, the limit load (48) approaches the Euler load.

It should be noted that for any fixing conditions at the edges, the transverse pressure on the surfaces has no effect on the buckling mode, which is identical to the classical mode.

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REFERENCES

1. A. E. Alekseev, "Nonlinear equations of elastic deformation of plates," *J. Appl. Mech. Tech. Phys.*, **43**, No. 3, 497–504 (2001).
2. A. E. Alekseev, "Linearized equations of nonlinear elastic deformation of thin plates," *J. Appl. Mech. Tech. Phys.*, **43**, No. 1, 133–139 (2002).
3. V. V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability*, Pergamon Press, Oxford (1963).
4. S. N. Korobeinikov, *Nonlinear Deformation of Solids* [in Russian], Izd. Sib. Otd. Ross. Akad. Nauk, Novosibirsk (2000).

5. H. Ziegler, *Principles of Structural Stability*, Blaisdell, New York (1968).
6. S. P. Timoshenko, *Stability of Elastic Systems* [in Russian], Gostekhteoretizdat, Moscow (1946).
7. A. D. Kerr and S. Tang, "The effect of lateral hydrostatic pressure on the instability of elastic solids, particularly beams and plates," *Trans. ASME, J. Appl. Mech.*, **11**, No. 5 (1965).
8. A. N. Guz', *Stability of Elastic Bodies under Uniform Compression* [in Russian], Naukova Dumka, Kiev (1979).
9. A. N. Guz', *Stability of Elastic Bodies under Large Strain* [in Russian], Naukova Dumka, Kiev (1973).